

UNIFORMLY BOUNDED REPRESENTATIONS FOR THE LORENTZ GROUPS

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Abstract. A family of uniformly bounded class 1 representations of the Lorentz groups is constructed. This family of representations includes, but is larger than, a similar family of representations constructed by Lipsman. The construction technique relies on a multiplicative analysis of various operators under a Mellin transform.

1. Introduction. Let $G = SO_e(1, n+1)$ ($n \geq 2$) be one of the Lorentz groups. The class 1 principal series of representations of G may be regarded as a family of unitary representations $T(\cdot, s)$ for $s \in i\mathbb{R}$. Lipsman constructs a family of representations $R(\cdot, s)$ for $-1 < \operatorname{Re} s < 1$ with the following properties:

- (1) $R(\cdot, s)$ is unitarily equivalent to $T(\cdot, s)$ for $s \in i\mathbb{R}$.
- (2) $\sup_{g \in G} \|R(g, s)\| < \infty$ for all s .
- (3) $s \rightarrow R(g, s)$ is an analytic operator-valued function for all $g \in G$.

The main result of this paper is the construction of a family of representations $R(\cdot, s)$ satisfying (1)–(3) for $-n/2 < \operatorname{Re} s < n/2$.

2. Multiplicative analysis of the Fourier transform on $\mathbb{R}^n$. Let $n \geq 2$ be fixed. Denote the standard inner product of vectors w and z in $\mathbb{R}^n$ by $w \cdot z$ and the length of a vector w by $|w| = (w \cdot w)^{1/2}$. Let $S = \{w \in \mathbb{R}^n : |w| = 1\}$. Identify $\mathbb{R}^n - \{0\}$ with the Cartesian product $\mathbb{R}^+ \times S$ where $\mathbb{R}^+$ is the multiplicative group of positive real numbers. Denote the usual Lebesgue measures on $\mathbb{R}^n$ and S by dw and $d\xi$. Define a measure d^*w on $\mathbb{R}^n - \{0\}$ by $d^*w = dw/|w|^n$. If dx is Lebesgue measure on the real line, then dx/x is a Haar measure for $\mathbb{R}^+$. Let $L_2(\mathbb{R}^n)$, $L_2(\mathbb{R}^n - \{0\})$, $L_2(S)$ and $L_2(\mathbb{R}^+)$ be the spaces of square integrable functions defined by these measures. By a classical result,

$$(2.1) \quad L_2(S) = \sum_{k=0}^{\infty} \bigoplus H_k$$

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where H_k is the space of spherical harmonics of order k . We may identify $L_2(\mathbf{R}^n - \{0\})$ with the Hilbert space tensor product $L_2(\mathbf{R}^+) \otimes L_2(S)$ and hence with

$$\sum_{k=0}^{\infty} \oplus (L_2(\mathbf{R}^+) \otimes H_k).$$

For f a function on $\mathbf{R}^n$, define $\mathcal{U}f$ on $\mathbf{R}^n - \{0\}$ by

$$(2.2) \quad \mathcal{U}f(w) = |w|^{n/2} f(w).$$

$\mathcal{U}$ is then an isometry of $L_2(\mathbf{R}^n)$ onto $L_2(\mathbf{R}^n - \{0\})$. For

$$f \in L_1(\mathbf{R}^n - \{0\}) \cap L_2(\mathbf{R}^n - \{0\})$$

define $\tilde{f}$ on $i\mathbf{R} \times S$ by

$$(2.3) \quad \tilde{f}(t, \xi) = \int_0^\infty x t f(x, \xi) \frac{dx}{x}.$$

The mapping $f \rightarrow \tilde{f}$ may be viewed as the tensor product of the identity mapping on $L_2(S)$ with the group theoretic Fourier transform mapping integrable functions on $\mathbf{R}^+$ to functions on the dual group $\hat{\mathbf{R}}^+ = i\mathbf{R}$. By the Plancherel theorem for locally compact abelian groups, this tensor product of maps has a unique extension to a mapping $\mathcal{M}$ from $L_2(\mathbf{R}^+) \otimes L_2(S)$ onto $L_2(i\mathbf{R}) \otimes L_2(S)$. Up to a scalar multiple, $\mathcal{M}$ is an isometry which will be referred to as the *Mellin transform* on $L_2(\mathbf{R}^n - \{0\})$.

Let $\mathcal{J}$ be the unitary operator on $L_2(\mathbf{R}^n - \{0\})$ defined by $(\mathcal{J}f)(x, \xi) = f(1/x, \xi)$. Let $\mathcal{F}$ denote the Fourier transform on $L_2(\mathbf{R}^n)$. For $f \in L_1(\mathbf{R}^n) \cap L_2(\mathbf{R}^n)$, $\mathcal{F}f$ is defined by

$$(2.4) \quad (\mathcal{F}f)(w) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbf{R}^n} f(z) e^{-i w \cdot z} dz.$$

Set $\hat{\mathcal{F}} = \mathcal{J} \mathcal{U} \mathcal{F} \mathcal{U}^{-1}$ and $\tilde{\mathcal{F}} = \mathcal{M} \hat{\mathcal{F}} \mathcal{M}^{-1}$. Then $\hat{\mathcal{F}}$ and $\tilde{\mathcal{F}}$ are unitary operators on $L_2(\mathbf{R}^+) \otimes L_2(S)$ and $L_2(i\mathbf{R}) \otimes L_2(S)$, respectively. The following lemma provides the desired multiplicative analysis of $\mathcal{F}$.

LEMMA 1. For $\tilde{f} \in L_2(i\mathbf{R}) \otimes H_k$ ($k=0, 1, 2, \dots$),

$$(2.5) \quad \tilde{\mathcal{F}} \tilde{f}(t, \xi) = \omega(t, k) \tilde{f}(t, \xi)$$

where

$$(2.6) \quad \omega(t, k) = i^k \frac{2^{-t} \Gamma((k+n/2-t)/2)}{\Gamma((k+n/2+t)/2)}.$$

Proof. Let φ be in H_k ($k=0, 1, 2, \dots$) and g a C^∞ function on $\mathbf{R}^+$ with support in a compact subset of the interval $(0, \infty)$. Since linear combinations of functions of the form $f=g\varphi$ are dense in $L_2(\mathbf{R}^+) \otimes H_k$, it suffices to show that formula (2.5) holds for the function $\tilde{f}=\mathcal{M}f$. Let $\hat{f}=\hat{\mathcal{F}}f$. It follows from Theorem (2.6.1) in Bochner [1, p. 38] that

$$(2.7) \quad \hat{f}(x, \xi) = i^k g_1(x) \varphi(\xi)$$

where

$$(2.8) \quad g_1(x) = \int_0^\infty g(yx) J_{k+(n-2)/2}(y) dy.$$

Set $\nu = k + (n-2)/2$ for convenience in notation. J_ν is the Bessel function of order ν . Now define

$$(2.9) \quad \tilde{g}_1(z) = \int_0^\infty x^z g_1(x) \frac{dx}{x} = \int_0^\infty \int_0^\infty x^z g(yx) J_\nu(y) dy \frac{dx}{x}.$$

It can be readily verified that the integral in (2.9) converges absolutely for $\operatorname{Re} z < \nu + 1$ and defines an analytic function in this domain. From Titchmarsh [5, p. 182],

$$(2.10) \quad \int_0^\infty x^\alpha J_\nu(x) \frac{dx}{x} = \frac{2^{\alpha-1} \Gamma((\nu+\alpha)/2)}{\Gamma((\nu-\alpha+2)/2)}$$

for $-\nu < \operatorname{Re} \alpha < \frac{1}{2}$. For $\frac{1}{2} < \operatorname{Re} z < \nu + 1$ interchange of the order of integration in (2.9) is valid and yields the result

$$(2.11) \quad \tilde{g}_1(z) = \frac{2^{-z} \Gamma((\nu+1-z)/2)}{\Gamma((\nu+1+z)/2)} \tilde{g}(z)$$

where

$$\tilde{g}(z) = \int_0^\infty x^z g(x) \frac{dx}{x}.$$

Since both sides of (2.11) define analytic functions of z in the domain $\operatorname{Re} z < \nu + 1$, the result remains valid for $z = t \in i\mathbf{R}$. Equation (2.5) now follows from (2.7) and (2.11). This completes the proof of the lemma.

3. Mellin transforms of A and B operators. For $s \in i\mathbf{R}$, let $B(s)$ be the unitary operator on $L_2(\mathbf{R}^n)$ defined by

$$\begin{aligned} B(s)f(w) &= |w|^{-s}f(w) & \text{for } w \neq 0, \\ &= 0 & \text{for } w = 0, \end{aligned}$$

and set $A(s) = \mathcal{F}^{-1}B(s)\mathcal{F}$.

This section is devoted to establishing a multiplicative analysis of the operators $A(s)$ and $B(s)$. Prior to this, estimates on the growth of the quotient of gamma functions in Lemma 1 must be obtained.

LEMMA 2. For $c \in \mathbf{R}^+$ and z in the strip $-c < \operatorname{Re} z < c$, define

$$F(c, z) = \frac{\Gamma((c+z)/2)}{\Gamma((c-z)/2)}.$$

There exists a constant C such that whenever $c > 0$, $\varepsilon > 0$ and $z = x + iy$ with $|x| < c - \varepsilon$, then

$$(3.1) \quad |F(c, z)| \leq Ce^{3|x|} 2^{-x} (1 + 2|x|/\varepsilon) |c + 2 + iy|^x.$$

Proof. The proof is modeled after the argument given by Kunze and Stein [3, p. 758] to prove a similar result. From Stirling's formula, for $\alpha > 0$ and $w = u + iv \in \mathbb{C}$ with $|u| < \alpha$,

$$(3.2) \quad \log \frac{\Gamma(\alpha + w)}{\Gamma(\alpha - w)} = (\alpha + w - \tfrac{1}{2}) \log \frac{\alpha + w}{\alpha + iv} - (\alpha - w - \tfrac{1}{2}) \log \frac{\alpha - w}{\alpha - iv} \\ - 2w + i(2\alpha - 1) \arg(\alpha + iv) + 2w \log |\alpha + iv| + B(w)$$

where $B(w)$ is a bounded function. For $\alpha > \frac{1}{2}$ and $|u| < \alpha - \frac{1}{2}$, routine estimates show that

$$(3.3) \quad |(\alpha \pm w - \tfrac{1}{2}) \log((\alpha \pm w)/(\alpha \pm iv))| \leq 2|u|.$$

It follows from (3.2) and (3.3) that there exists a constant C independent of α such that

$$(3.4) \quad |\Gamma(\alpha + w)/\Gamma(\alpha - w)| \leq Ce^{6|u|} |\alpha + iv|^{2u} \quad \text{for } |u| < \alpha - \tfrac{1}{2}.$$

Now let $c > 0$, $\varepsilon > 0$ and $z = x + iy$ with $|x| < c - \varepsilon$. Replacing α by $(c + 2)/2$ and w by $z/2$ in (3.4), we obtain the estimate

$$(3.5) \quad |F(c, z)| = \left| \frac{c - z}{c + z} \right| \left| \frac{\Gamma((c + 2 + z)/2)}{\Gamma((c + 2 - z)/2)} \right| \\ \leq \left| \frac{c - z}{c + z} \right| Ce^{3|x|} \left| \frac{c + 2 + iy}{2} \right|^x \leq C \left(1 + \frac{2|x|}{\varepsilon} \right) e^{3|x|} 2^{-x} |c + 2 + iy|^x.$$

LEMMA 3. Let $s = \sigma_0 + i\tau_0$ with $|\sigma_0| < n/2 - \varepsilon$. Set $C_n = C^2 e^{3n} 2^n$. For ω as in Lemma 1,

$$\sup_{k \in \mathbb{Z}^+; t \in i\mathbb{R}} |\omega(t + s, k) \omega(t - s, k)| \leq C_n (1 + n/\varepsilon)^2 (1 + 4/n |\tau_0|)^{n/2}.$$

Proof. From Lemma 2,

$$\sup_{k \in \mathbb{Z}^+; t = t_0 + i\tau} |\omega(t + s, k) \omega(t - s, k)| = \sup_{k, \tau} |F(k + n/2, -s - t) F(k + n/2, s - t)| \\ \leq C_n (1 + n/\varepsilon)^2 \sup_{k, \tau} \left| \frac{k + n/2 + i(\tau_0 - \tau)}{k + n/2 - i(\tau_0 + \tau)} \right|^{\sigma_0} \\ \leq C_n (1 + n/\varepsilon)^2 \sup_{k, \tau} \left| 1 + \frac{2|\tau_0|}{|k + n/2 - i\tau|} \right|^{| \sigma_0 |} \\ \leq C_n (1 + n/\varepsilon)^2 (1 + 4/n |\tau_0|)^{n/2}.$$

THEOREM 1. (1) $s \rightarrow A(s)B(s)$ is a commutative family of operators for $s \in i\mathbb{R}$.

(2) There exists an operator-valued function $s \rightarrow C(s)$ defined on the strip $-n/2 < \operatorname{Re} s < n/2$ such that

$$C(s) = A(s)B(s)A(-s)B(-s) \quad \text{for } \operatorname{Re} s = 0.$$

(3) $\|C(s)\| \leq C_n (1 + n/\varepsilon)^2 (1 + 4/n |\operatorname{Im} s|)^{n/2}$ for $\varepsilon > 0$ and $|\operatorname{Re} s| < n/2 - \varepsilon$.

Proof. Let $\tilde{B}(s)$ and $\tilde{A}(s)$ be the unitary operators on $L_2(i\mathbb{R}) \otimes L_2(S)$ defined by

$$\tilde{B}(s) = (\mathcal{M}\mathcal{U})B(s)(\mathcal{M}\mathcal{U})^{-1}, \quad \tilde{A}(s) = (\mathcal{M}\mathcal{U})A(s)(\mathcal{M}\mathcal{U})^{-1}.$$

It is easily checked that for $f \in L_2(i\mathbf{R}) \otimes L_2(S)$,

$$(3.6) \quad \tilde{B}(s)f(t, \xi) = f(t-s, \xi),$$

$$(3.7) \quad \tilde{A}(s) = \mathcal{F}^{-1}\tilde{B}(-s)\mathcal{F}.$$

From (3.6), (3.7) and Lemma 1, it follows that for $f \in L_2(i\mathbf{R}) \otimes H_k$ ($k=0, 1, 2, \dots$),

$$(3.8) \quad \tilde{A}(s)\tilde{B}(s)f(t, \xi) = (\omega(t+s, k)/\omega(t, k))f(t, \xi).$$

(1) is an immediate consequence of (3.8). Now for $|\operatorname{Re} s| < n/2$, define $\tilde{C}(s)$ on $L_2(i\mathbf{R}) \otimes H_k$ by

$$(3.9) \quad \tilde{C}(s)f(t, \xi) = \frac{\omega(t+s, k)}{\omega(t, k)} \frac{\omega(t-s, k)}{\omega(t, k)} f(t, \xi).$$

From Lemma 3, the multiplier in (3.9) is a bounded function of $t \in i\mathbf{R}$ and $k \in \mathbf{Z}^+$ for each s in the strip $|\operatorname{Re} s| < n/2$. Hence (3.9) defines a bounded operator on $L_2(i\mathbf{R}) \otimes L_2(S)$. Clearly $s \rightarrow \tilde{C}(s)$ is analytic. Defining $C(s) = (\mathcal{M}\mathcal{U})^{-1}\tilde{C}(s)\mathcal{M}\mathcal{U}$, we obtain an analytic operator-valued function on $L_2(\mathbf{R}^n)$ satisfying (2) and (3).

4. Construction of uniformly bounded representations for the Lorentz groups.

For $n \geq 1$, define $G = SO_e(1, n+1)$ to be the connected component of the identity in the group of $(n+2) \times (n+2)$ real matrices g for which ${}^t g p_0 g = p_0$ where ${}^t g$ is the transpose of g and

$$p_0 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -I_n & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

The family of *Lorentz groups* is the collection $\{SO_e(1, n+1) : n \geq 1\}$. For $n=1$, the group $SO_e(1, 2)$ is locally isomorphic to $SL(2, \mathbf{R})$. Uniformly bounded representations of this group are constructed by Kunze and Stein [2]. A modified version of the Mellin transform analysis used here applies to this group but yields no new representations. We shall therefore assume $n \geq 2$ in the remainder of this paper.

Let M , A , N and V be the subgroups of G defined as follows:

$$\begin{aligned} A &= \left\{ a = a(x) = \begin{bmatrix} x & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & 1/x \end{bmatrix} : x \in \mathbf{R}^+ \right\}, \\ M &= \left\{ m = m(h) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & 1 \end{bmatrix} : h \in SO(n, \mathbf{R}) \right\}, \\ N &= \left\{ u = u(w) = \begin{bmatrix} 1 & w & \frac{1}{2}|w|^2 \\ 0 & I_n & {}^t w \\ 0 & 0 & 1 \end{bmatrix} : w \in \mathbf{R}^{1 \times n} \right\}, \\ V &= \{v = v(w) = {}^t u(w)\}. \end{aligned}$$

Set $B = MAN$. For n odd, set $J_n = -I_n$ and for n even, set

$$J_n = \begin{bmatrix} -1 & & & & \\ & -1 & & & \\ & & \ddots & & \\ & & & -1 & \\ & & & & 1 \end{bmatrix}.$$

Now set

$$p_n = \begin{bmatrix} 0 & 0 & 1 \\ 0 & J_n & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Throughout the following discussion, n will be fixed and we write J for J_n and p for p_n . Then p is a representative of the nontrivial element of the Weyl group for the rank 1 semisimple Lie group G . B is a closed subgroup of G and by the Bruhat Lemma, $G = B \cup BpB = B \cup BVp$. Set $\tilde{B} = p^{-1}Bp$. Then $\tilde{B} = MAV$ and $G = \tilde{B} \cup \tilde{B}p\tilde{B}$. The set BV is a dense set of full measure in G .

For $s \in i\mathbf{R}$, let $T(\cdot, s)$ be the unitary representation of G induced from the character $ma(x)u \rightarrow x^s$ of B . The family of representations $T(\cdot, s)$ is the class 1 principal series of G . These representations may be realized as operators on $L_2(V)$ defined by the formulae

$$(4.1) \quad \begin{aligned} T(g, s)f(v) &= x^{n/2+sf(v \cdot g)} & \text{if } vg = (ma(x)u)(v \cdot g) \in BV, \\ &= 0 & \text{if } vg \notin BV. \end{aligned}$$

Since $w \rightarrow v(w)$ is a measure preserving homeomorphism of $\mathbf{R}^n$ onto V , we may identify $L_2(V)$ with $L_2(\mathbf{R}^n)$ by writing $f(w)$ for $f(v(w))$. Note that

$$(4.2) \quad v(w)a(x) = a(x)v(xw) \quad \text{for all } (x, w) \in \mathbf{R}^+ \times \mathbf{R}^n$$

and

$$(4.3) \quad v(w)m(h) = m(h)v(wh) \quad \text{for all } (h, w) \in SO(n, \mathbf{R}) \times \mathbf{R}^n.$$

In (4.3) wh is to be interpreted as the product of the row matrix w and the orthogonal matrix h . From (4.1)–(4.3), it follows that the operators $T(g, s)$ for $g = m(h)a(x)v(w_0) \in \tilde{B}$ are defined on $L_2(\mathbf{R}^n)$ by

$$(4.4) \quad T(m(h)a(x)v(w_0), s)f(w) = x^{n/2+sf(xwh+w_0)}.$$

An easy matrix calculation shows that for $w \neq 0$,

$$v(w)p = m(J - 2^t w w J / |w|^2) a(2/|w|^2) u(-wJ) v(2wJ/|w|^2).$$

For $f \in L_2(\mathbf{R}^n)$ it follows that

$$(4.5) \quad \begin{aligned} T(p, s)f(w) &= (2/|w|^2)^{n/2+sf(2wJ/|w|^2)} & \text{for } w \neq 0, \\ &= 0 & \text{if } w = 0. \end{aligned}$$

Since $G = \tilde{B} \cup \tilde{B}p\tilde{B}$, the representation $T(\cdot, s)$ is completely determined by (4.4) and (4.5). Now define $R(\cdot, s) = A(s)T(\cdot, s)A(-s)$.

LEMMA 4. $R(\cdot, s)|_{\tilde{B}} = T(\cdot, 0)|_{\tilde{B}}$.

Proof. Since $A(s) = \mathcal{F}^{-1}B(s)\mathcal{F}$, it suffices to show that

$$(4.6) \quad B(s)\hat{T}(g, s) = \hat{T}(g, 0)B(s)$$

for all $g \in \tilde{B}$ where $\hat{T}(g, s) \equiv \mathcal{F}T(g, s)\mathcal{F}^{-1}$. Let $g = m(h)a(x)v(w_0)$. It follows easily from (4.4) that

$$(4.7) \quad \hat{T}(g, s)f(w) = x^{-n/2+s}e^{tw_0 \cdot x^{-1}wh}f(x^{-1}wh).$$

Formula (4.6) is an immediate consequence of (4.7).

LEMMA 5. $A(2s)T(\cdot, s) = T(\cdot, -s)A(2s)$.

Proof. From (4.7) it follows easily that

$$B(2s)\hat{T}(g, s) = \hat{T}(g, -s)B(2s) \quad \text{for all } g \in \tilde{B}.$$

Hence it suffices to show that

$$(4.8) \quad A(2s)T(p, s) = T(p, -s)A(2s).$$

Let $\tilde{T}(p, s) = (\mathcal{M}\mathcal{U})T(p, s)(\mathcal{M}\mathcal{U})^{-1}$. For $f \in L_2(i\mathbf{R}) \otimes L_2(S)$, it follows from (4.5) that

$$(4.9) \quad \tilde{T}(p, s)f(t, \xi) = 2^{t-s}f(2s-t, \xi J).$$

Note that if $\xi \rightarrow g(\xi)$ is a function in H_k , then $\xi \rightarrow g(\xi J)$ is also a function in H_k . Hence the subspaces $L_2(i\mathbf{R}) \otimes H_k$ ($k=0, 1, 2, \dots$) are invariant under $\tilde{T}(p, s)$. From (2.5), (3.7) and (4.8), it follows that for $f \in L_2(i\mathbf{R}) \otimes H_k$,

$$(4.10) \quad \tilde{A}(2s)\tilde{T}(p, s)f = \tilde{T}(p, -s)\tilde{A}(2s)f.$$

Formula (4.8) now follows easily from (4.10).

LEMMA 6. (1) $R(\cdot, s) = R(\cdot, -s)$ for all $s \in i\mathbf{R}$.

(2) $R(p, s) = A(s)B(s)A(-s)B(-s)T(p, 0)$ for all $s \in i\mathbf{R}$.

Proof. (1)

$$\begin{aligned} R(\cdot, s) &= A(s)T(\cdot, s)A(-s) = A(-s)A(2s)T(\cdot, s)A(-2s)A(s) \\ &= A(-s)T(\cdot, -s)A(s) = R(\cdot, -s). \end{aligned}$$

(2) Formula (4.5) implies that $T(p, s) = 2^s B(2s)T(p, 0)$. From Lemma 5 and the fact that $T(p, 0)^2$ is the identity, it follows that

$$(4.11) \quad T(p, 0)A(\cdot, -s)T(p, 0) = 2^{-s}B(-s)A(-s)B(-s).$$

Hence

$$\begin{aligned} R(p, s) &= 2^s A(s)B(2s)T(p, 0)A(-s)T(p, 0)^2 \\ &= A(s)B(s)A(-s)B(-s)T(p, 0). \end{aligned}$$

THEOREM 2. *For each $g \in G$, the function $s \rightarrow R(g, s)$ initially defined for $s \in i\mathbb{R}$ has an extension to an analytic function on the strip $D = \{s \in \mathbb{C} : |\operatorname{Re} s| < n/2\}$. The resulting operators $R(g, s)$ have the property that $R(g, s) = R(g, -s)$ for all $g \in G$ and $s \in D$. For all $s \in D$, $g \rightarrow R(g, s)$ is a uniformly bounded representation of G .*

Proof. Let $s \in D$. For $g \in \tilde{B} = MAV$, define $R(g, s) = T(g, 0)$. Define $R(p, s) = C(s)T(p, 0)$ where $C(s)$ is the operator-valued function defined in Theorem 1. If $g \in G$ is of the form $g = g_1 p g_2$ for g_1 and g_2 in $\tilde{B}$, set $R(g, s) = R(g_1, s)R(p, s)R(g_2, s)$. By the preservation of functional equations under analytic continuation and the fact that $G = \tilde{B} \cup \tilde{B}p\tilde{B}$, it follows that the operators $R(g, s)$ are well defined for all $g \in G$ and $s \in D$ and satisfy the symmetry condition $R(g, s) = R(g, -s)$. Moreover, $g \rightarrow R(g, s)$ is a representation of G for all $s \in D$ and $s \rightarrow R(g, s)$ is analytic for all $g \in G$. Since $\sup_{g \in G} \|R(g, s)\| = \|R(p, s)\| = \|C(s)\|$, the representations $R(\cdot, s)$ are uniformly bounded.

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